Winter 2017 MATH 15910 Section 55

Exam 1 Solution

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1. $A \subset X$, Prove $A \cap X = A$.

Proof.

Let $x \in A \cap X = \{u \mid u \in A \text{ and } u \in X\}$.

By def of intersection, $x \in A$.

Let $y \in A$ $\because A \subset X$

$\therefore y \in X$ by def of subset

$\therefore y \in A$ and $y \in X$

$\therefore$ By def of intersection,

$y \in A \cap X$.

Thus, $A \cap X = A$.

2. 1) $X = \{55, 2017\}$.

List of elements of power set $P(X)$ of X.

List of elements of $P(X)$:

$\emptyset$, $\{55\}$, $\{2017\}$, $\{55, 2017\}$.

Btw, $P(X) = \{\emptyset, \{55\}, \{2017\}, \{55, 2017\}\}$

2) $X = \emptyset$. List of elements of the power set $P(X)$ of X.

List of elements of $P(X)$:

$\emptyset$.

Note

$(P(X) = \{\emptyset\})$.

It can be proved that $\emptyset \subset \emptyset$

(by "vacuous truth".)

$$\underline{\forall x \in \emptyset}, \ \underbrace{prop(x)}_{\text{some proposition}}$$

This implies $prop(x)$ is true because there is no way to falsify the statement - given $\underline{x \in \emptyset}$ will never be true.

It can also be proved that

$\forall A \neq \emptyset$, $A \not\subset \emptyset$.

3. $1+2+\cdots+n = \frac{n(n+1)}{2}$

Proof.

We prove by induction on $n \in \mathbb{N}$.

Base case: $n=1$

$1 = \frac{1(1+1)}{2}$ ✓

Inductive hypothesis:

For a fixed n, $1+2+\cdots+n = \frac{n(n+1)}{2}$

WTS: $1+2+\cdots+n+(n+1) = \frac{(n+1)((n+1)+1)}{2}$

Now, consider

$1+2+\cdots+n+(n+1)$

$= \frac{n(n+1)}{2} + (n+1)$ (by I.H.)

$= \frac{n^2+n}{2} + \frac{2n+2}{2}$

$= \frac{n^2+3n+2}{2}$

$= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}$.

By Principles of Mathematical Induction, the proposition holds.

4. 1) Relation on a set X.

Def 1.4.2 A relation on a set X is a subset of $X \times X = \{(a,b) \mid a \in X \text{ and } b \in X\}$.

2) Equivalent relation on a set X.

An equivalent relation on X is a relation R on X s.t.

(ER1) (Reflexive)

$\forall a \in X$, $(a,a) \in R$.

(ER2) (Symmetric)

$\forall a,b \in X$, if $(a,b) \in R$, then $(b,a) \in R$.

(ER3) (Transitive)

$\forall a,b,c \in X$, if (a,b) and $(b,c) \in R$, then $(a,c) \in R$.

3) Def of $\mathbb{Z}_2$

$\mathbb{Z}_2$ is the set of equivalent classes $\overline{0}$ and $\overline{1}$.

(For $\overline{a}, \overline{b} \in \mathbb{Z}_2$, $\overline{a}+\overline{b} = \overline{a+b}$ and $\overline{a}\,\overline{b} = \overline{ab}$)

function
5. $f: \mathbb{Z} \to \mathbb{Z}$. $f(n) = n+1$. Prove f bijection.

<u>Proof.</u>

<u>Pf of surjection</u>

Let $y \in \mathbb{Z}$. Consider $x = y-1$. $\in \mathbb{Z}$

Then $x+1 = (y-1)+1 = y$.

$\Rightarrow f(x) = y$ where $x \in \mathbb{Z}$.

$\Rightarrow y \in f(\mathbb{Z})$

Moreover, $\forall z \in f(\mathbb{Z})$, $z \in \mathbb{Z}$ given $f: \mathbb{Z} \to \mathbb{Z}$.

$\therefore f(\mathbb{Z}) = \mathbb{Z}$

$\therefore f$ is surjective

<u>Pf of injection</u>

Let $x, x' \in \mathbb{Z}$ s.t. $f(x) = f(x')$

$\Rightarrow x+1 = x'+1 \quad \Rightarrow x = x'$

$\therefore f$ is injective

$\therefore f$ is surjective and injective

$\therefore f$ is bijective

6. $f: A \to B$, $B_1, B_2 \subset B$. Prove

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$

<u>Proof.</u>

Consider $x \in f^{-1}(B_1 \cup B_2)$.

$\Leftrightarrow f(x) \in B_1 \cup B_2$,

$\Leftrightarrow f(x) \in B_1$ or $f(x) \in B_2$.

$\Leftrightarrow x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$.

$\Leftrightarrow x \in f^{-1}(B_1) \cup f^{-1}(B_2)$

(Note: $\Leftrightarrow$ means iff
$\Updownarrow$
if and only if)

We have proved that

$\forall x \in f^{-1}(B_1 \cup B_2)$, $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$,

and $\forall y \in f^{-1}(B_1) \cup f^{-1}(B_2)$, $y \in f^{-1}(B_1 \cup B_2)$.

Thus, $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$