Winter 2017 MATH 15910 Section 55

HW4 Solution

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1. Prove $\sqrt{2} \notin \mathbb{Q}$

Proof.

WTS: $\nexists\, x \in \mathbb{Q}$ s.t. $x^2 = 2$

Suppose not. Then $\exists\, p, q \in \mathbb{Z}, q \neq 0$,

s.t. $(\frac{p}{q})^2 = 2$, and

WLOG, p, q have no common factor.

$\Rightarrow p^2 = 2q^2$

$\Rightarrow p^2$ is even integer

$\Rightarrow p$ is even

$\Rightarrow p = 2r$ for some $r \in \mathbb{Z}$

$\therefore (2r)^2 = 2q^2$

$\therefore q^2 = 2r^2$

$\Rightarrow q^2$ even

$\Rightarrow q$ even

$\Rightarrow\!\times\!\Leftarrow$

(contradicting the assumption that p, q have no common factor)

2. Ex 3.1.12

Let n be a positive integer, not a perfect square.

Let $A = \{x \in \mathbb{Q} \mid x^2 < n\}$. Show A is bounded in $\mathbb{Q}$, but has neither greatest lower bound nor least upper bound in $\mathbb{Q}$.

Conclude $\sqrt{n}$ exists in $\mathbb{R}$.
(That is, $\exists\, a \in \mathbb{R}$ s.t. $a^2 = n$)

Proof.

① Let $n' \in \mathbb{Z}$, s.t. $n' > n$ and n' is a perfect square.

$\Rightarrow \exists\, n'' \in \mathbb{Z}, n'' > 0$ s.t. $n''^2 = n'$.

It is easy to show that

$n'' > x, \forall x \in A$,

$-n'' < x, \forall x \in A$.

$\Rightarrow A$ is bounded in $\mathbb{Q}$.

② Suppose A has sup (lub) in $\mathbb{Q}$.

$\exists\, p, q \in \mathbb{Z}, q \neq 0, p > 0$, s.t. $\sup A = \frac{p}{q}$

$\because \sup A$ is upper bound of A

$\therefore (\frac{p}{q})^2 > n$.

Let $m = \frac{kp}{q} = k(\frac{p}{q})$ where $k \in \mathbb{Q}, k > 0$.

$\therefore (\frac{p}{q} - m)^2 = (1-k)^2 (\frac{p}{q})^2$

We can select $k \in \mathbb{Q}$ s.t. $(\frac{p}{q} - m)^2 < (\frac{p}{q} - n)^2$

(cont'd).

But now we have $z = \frac{p}{q} - m$

$= \sqrt{(\frac{p}{q} - m)^2}$

$= \sqrt{(1-k)^2}\sqrt{(\frac{p}{q})^2}$

$= (1-k)(\frac{p}{q}) \in \mathbb{Q}$

s.t. $z > n$, $z < \sup A = \frac{p}{q}$.

We have

$z > x$, $\forall x \in A$

but $z < \sup A$.

This is a contradiction (by def of least upper bound).

Note: sup $\Leftrightarrow$ supremum

$\Leftrightarrow$ lub

$\Leftrightarrow$ least upper bound

Similar for greatest lower bound.

③

For existence of $\sqrt{n}$, given A is bounded, and nonempty, by least upper bound property,

$\sup A$ exists. Call it a. $a \in \mathbb{R}$.

WTS: $a^2 = n$.

Suppose for contradiction $a^2 \neq n$.

1) $a^2 < n$.

Let $\varepsilon \in (0,1)$ and $\varepsilon < \frac{n-a^2}{2a+1}$,

$(a+\varepsilon)^2 = a^2 + 2\varepsilon a + \varepsilon^2$

$\leq a^2 + 2\varepsilon a + \varepsilon$

$= a^2 + \varepsilon(2a+1)$

$< n$

But this means $\exists p \in \mathbb{Q}$. $p^2 \in ((a+\varepsilon)^2, n)$, and $p \in A$.

$\Rightarrow\Leftarrow$

2) Similar for the $a^2 > n$ case

$\therefore \exists a \in \mathbb{R}$ s.t. $a^2 = n$

Ex 3.1.14

(i) $\text{lub}(A \cup B) = \max\{\text{lub}(A), \text{lub}(B)\}$

Proof. I use notation sup instead of lub here. I apologize for this confusion

Suppose first $\sup A \geq \sup B$.

Then $\forall a \in A$, $a \leq \sup A$

$\forall b \in B$, $b \leq \sup B \leq \sup A$

$\Rightarrow \forall y \in A \cup B$, $y \leq \sup A$

$\therefore \sup A$ is upper bound for $A \cup B$. (*)

WTS: $\sup A$ is least upper bound of $A \cup B$

Let u be an arbitrary upper bound of $A \cup B$.

Then u is an upper bound of A.

$\because \sup A$ is least upper bound of A by def

$\therefore u \geq \sup A$ (**)

By (*)(**), $\sup(A \cup B) = \sup A$.

If $\sup B \geq \sup A$, Similarly, we have $\sup(A \cup B) = \sup B$.

Thus, $\sup(A \cup B) = \max(\sup A, \sup B)$.

(ii) If $A+B=\{a+b \mid a\in A,\ b\in B\}$, then $\sup(A+B)=\sup A+\sup B$.

Proof.

First, WTS: $\sup A+\sup B$ is upper bound for set $A+B$.

By def, $\forall a\in A,\ a\le \sup A$

$\forall b\in B,\ b\le \sup B$.

$\Rightarrow \forall a\in A,\ b\in B,\ a+b\le \sup A+\sup B$

By def of set $A+B$, any element in $A+B$ can be written as $a+b$ for some $a\in A,\ b\in B$.

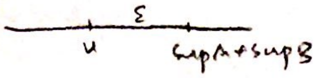
$\Rightarrow$ $\sup A+\sup B$ is upper bound of set $A+B$.

Now, WTS: $\sup A+\sup B$ is least upper bound for set $A+B$.

Let u be an arbitrary upper bound of $A+B$.

Suppose $u<\sup A+\sup B$

Let $\varepsilon=\sup A+\sup B-u$



Select $a'\in A$ s.t. $a'>\sup A-\frac{\varepsilon}{2}$

Select $b'\in B$ s.t. $b'>\sup B-\frac{\varepsilon}{2}$

$\Rightarrow a'+b'>\sup A+\sup B-\varepsilon=u$

But $a'+b'\in A+B$.

$\Rightarrow\Leftarrow$

4. Prove $\mathbb{Z}$ is not dense in $\mathbb{R}$

Def $A\subset\mathbb{R}$ is dense in $\mathbb{R}$ if for any pair of real numbers a,b w/ $a<b$, $\exists r\in A$ s.t. $a<r<b$

("dense" in our textbook)

Let $a=\frac{1}{3}$, $b=\frac{1}{2}$, we have $a<b$.

But $\nexists r\in\mathbb{Z}$ s.t. $a<z<b$.

Done.