Winter 2017 MATH 15900 Section 55

HW5 Solution

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Ex 3.2.9

(i) Any irrational number multiplied by any non-zero rational number is irrational

Proof. Assume $x \in \mathbb{R} \setminus \mathbb{Q}$, $a = \frac{p}{q} \in \mathbb{Q}$
$p, q \neq 0$.
$p, q \in \mathbb{Z}$

$$a x = \frac{px}{q}$$

Assume $ax \in \mathbb{Q}$, then $\exists\, r, s \in \mathbb{Z}$, $s \neq 0$ s.t. $\frac{px}{q} = \frac{r}{s} \Rightarrow x = \frac{rq}{ps} \in \mathbb{Q}$

$\Rightarrow\Leftarrow$ contradiction given $x \in \mathbb{R} \setminus \mathbb{Q}$

(iii) Product of two irrational number rational or irrational

① $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$
$\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbb{Q}$

② $\sqrt{2} + 1 \in \mathbb{R} \setminus \mathbb{Q}$, $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$
$(\sqrt{2} + 1) \cdot \sqrt{2} = 2 + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

Ex 3.6.5

Limit of convergent sequence is unique.

Proof. Let $\{a_n\}$ be a sequence.

Suppose $a_n \to p$, $a_n \to p'$

WTS: $p = p'$

Let $\varepsilon > 0$

$\because a_n \to p$

$\therefore \exists\, N_1 \in \mathbb{N}$ s.t.

$n \geq N_1 \Rightarrow |a_n - p| < \frac{\varepsilon}{2}$

$\because a_n \to p'$

$\therefore \exists\, N_2 \in \mathbb{N}$ s.t.

$n \geq N_2 \Rightarrow |a_n - p'| < \frac{\varepsilon}{2}$

Let $N = \max(N_1, N_2)$

$\therefore n \geq N \Rightarrow |p - p'|$
$\leq |p - a_n| + |a_n - p'|$
$< \varepsilon$

where ε is an arbitrary positive number fixed by us.

$\therefore p = p'$.

Ex 3.6.13

(i) Prove Cauchy sequence in $\mathbb{R}$ is bdd.

Proof. Suppose $\{a_n\}$ is a Cauchy sequence in $\mathbb{R}$. Let $\varepsilon > 0$.

Then $\exists N \in \mathbb{N}$ s.t. $m, n \geq N \Rightarrow d(p_m, p_n) < \varepsilon$ $(*)$

If we let n in $(*)$ to be $N+1$, we have

if $m \geq N$, then $|p_m - p_{N+1}| < \varepsilon$.

Let $M = \max\left\{ \max_{1 \leq i \leq N} \{|p_i - p_{N+1}|\}, \varepsilon \right\} > 0$.

Then $\forall k \in \mathbb{N}$, $|p_k - p_{N+1}| < M$.

$\therefore \exists R$ s.t. $|p_k| < R$, $\forall k \in \mathbb{N}$.
(could show in more detail)

(ii) If $(a_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, show for $\forall \varepsilon > 0$, $\exists$ subsequence $(a_{k_j})_{j \in \mathbb{N}}$ s.t. $|a_{k_j} - a_{k_{j+1}}| < \frac{\varepsilon}{2^{j+1}}$ $\forall j \in \mathbb{N}$.

Proof.
$\because \{a_k\}$ is bdd

By Lemma 3.6.10

$\{a_k\}$ has convergent subsequence $\{b_{n_i}\}$. Suppose $b_{n_i} \to p$

$\therefore \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$n_j, n_{j+1} \geq N \Rightarrow$

$|b_{n_j} - p| < \frac{\varepsilon}{2^{j+2}}$

$|b_{n_{j+1}} - p| < \frac{\varepsilon}{2^{j+2}}$

Let b_{n_α} be the first b_{n_i}'s s.t. $n_i \geq N$. Let $a_{k_j} = b_{n_{(\alpha+j-1)}}$

We see that

given $k_j, k_{j+1} \geq n_\alpha \geq N$,

we have $|a_{k_j} - p| < \frac{\varepsilon}{2^{j+2}}$

$|a_{k_{j+1}} - p| < \frac{\varepsilon}{2^{j+2}}$

By triangle inequality, ---.

Thm 3.6.14

See text book

Ex 3.6.21

(i) Infinite subset of $\mathbb{R}$ that does not have an accumulation point in $\mathbb{R}$

$\mathbb{N}$. Explain.

(ii) Bdd subset of $\mathbb{R}$ that does not have an acc point in $\mathbb{R}$.

$\{0, 1\}$. Explain.

(iii) Bdd infinite subset of $\mathbb{Q}$ that does not have an acc pt in $\mathbb{Q}$.

Construct a sequence $\{a_n\}$ as follows.

Let $a_n \in (\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}) \cap \mathbb{Q}$ for $n \in \mathbb{N}$

1. Prove sequence $\{a_n\}$ converges to $\sqrt{2}$
2. Prove that $\sqrt{2}$ is an accumulation pt of the set $A = \{a_n\}_{n \in \mathbb{N}}$
3. Prove that q is not acc pt of A $\forall q \in \mathbb{Q}$
(by using the fact that there are infinitely many points within arbitrary distance ε to $\sqrt{2}$).